Compactness characterisations of large cardinals with strong Henkin models

Jonathan Osinski[(1)] Alejandro Poveda[(2,3)]

[(1)] Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

[(2)] Center of Mathematical Sciences and Applications, Harvard University, 20 Garden Street, Cambridge MA 02138, United States of America

[(3)] Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge MA 02138, United States of America

Abstract

We consider compactness properties for strong logics in terms of *strong Henkin models* and give characterisations of supercompact cardinals, $\mathrm{C}^{(n)}$-extendible cardinals, and Vopěnka's Principle by these properties. Moreover, we give a characterisation of superstrong cardinals in terms of compactness properties using the previously considered *weak Henkin models*.

1 Introduction

The well-known large cardinal notions of weakly compact and strongly compact cardinals are characterised in terms of compactness properties for infinitary languages. Other large cardinal notions can be characterised in terms of compactness properties for other strong languages.

In [8], Boney characterised strong cardinals by a compactness principle providing the existence of Henkin models of second-order theories. In [9], the authors introduced a notion we will call in this paper *weak Henkin models* of a general logic (cf. Definition 2) and used it to characterise Woodin cardinals in a similar way.

Being a weak Henkin model of a theory T is an in some sense unnatural notion: it includes reference to a model of set theory that includes a structure the model believes satisfies each sentence of T; but it does not need to contain T itself and so it cannot express this fact. This leads naturally to a strengthening of weak Henkin models that we will call *strong Henkin models* (cf. Definition 4).

In §§ 3 & 4, we will characterise the notions of supercompact cardinals and $\mathrm{C}^{(n)}$-extendible cardinals in terms of the strong Henkin model version of compactness, respectively (Theorems 5 & 10). The latter characterisation allows us to characterise Vopěnka's Principle (Corollary 12). In §5, we use a compactness principle about weak Henkin models to characterise superstrong cardinals (Theorem 15).

Corresponding author: Jonathan Osinski, `jonathan.osinski@uni-hamburg.de`

2 Preliminaries

2.1 Large cardinals

We will give the definitions of the relevant large cardinal notions and some relevant background.

A cardinal κ is *λ-strong* if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$. It is *strong* if it is λ strong for every $\lambda > \kappa$.

A cardinal κ is *Π_n-strong* if for every class A which is Π_n-definable without parameters and every λ there is an elementary embedding $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$ and $A \cap V_\lambda \subseteq A^M$ (cf. [5]).

A cardinal κ is *superstrong with target* λ if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $V_{j(\kappa)} \subseteq M$.

A cardinal κ is *λ-strongly compact* if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and such that there is $d \in M$ with $M \models |d| < j(\kappa)$ and $j``\lambda \subseteq M$. It is *strongly compact* if it is λ-strongly compact for every $\lambda > \kappa$.

A cardinal κ is *λ-supercompact* if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$. It is *supercompact* if it is λ-supercompact for every $\lambda > \kappa$. That κ is λ-supercompact is equivalent to the existence of a fine, normal, and κ-complete ultrafilter on $\mathcal{P}_\kappa\lambda$ (cf., e.g., [12, § 20]).

We write $C^{(n)}$ to denote the club class of ordinals α such that $V_\alpha \prec_{\Sigma_n} V$, i.e., such that V_α is an elementary substructure of the universe with respect to the Σ_n-formulas. Then, a cardinal κ is called $\mathrm{C}^{(n)}$*-extendible* if for every $\alpha > \kappa$ there is an elementary embedding $j : V_\alpha \to V_\beta$ such that $j(\kappa) > \alpha$ and $j(\kappa) \in \mathrm{C}^{(n)}$ (cf. [3]). Bagaria showed that extendible cardinals (cf. [13, p. 311]) are precisely the $\mathrm{C}^{(1)}$-extendible cardinals and that with growing n, the existence of a $\mathrm{C}^{(n)}$-extendible cardinal gains consistency strength. These cardinal notions stratify the large cardinal principle known as *Vopěnka's Principle* (VP; cf. [13, pp. 335–339]) in the following sense.

Theorem 1 (Bagaria; [3, Corollary 4.15])**.** VP holds if and only if for every n, there is a $\mathrm{C}^{(n)}$-extendible cardinal.

2.2 Abstract model theory

We make some remarks about the notions from abstract model theory we will use. As in first-order model theory, a *vocabulary* τ consists of finitary relation, function and constant symbols. Moreover, we will work with many sorted vocabularies, i.e., τ further contains a set of sort symbols. A τ-structure $\mathcal{A}$ has for every sort symbol s a domain A_s and furthermore interpretations of the relation, function, and constant symbols. A bijective map $f : \tau \to \sigma$ is called a *renaming* iff it restricts to respective bijections between the sets of sort, relation, function and constant symbols, all while respecting their respective arities. Notice that if $\mathcal{A}$ is a τ-structure and $f : \tau \to \sigma$ a renaming, then f induces a τ-structure $f(\mathcal{A})$ on the domain of $\mathcal{A}$.

Though we will mostly work with concretely given logics, some of our results contain statements that make reference to all logics simultaneously. In this case we refer to common definitions of logics given in abstract model theory (cf., e.g., [7, Chapter II]). We point out their in our context most important features. If τ is a vocabulary and $\mathcal{L}$ a logic, we write $\mathcal{L}[\tau]$ for the collection of $\mathcal{L}$-sentences over τ. We assume that $\mathcal{L}[\tau]$ is always a set (as opposed to a proper class). If $T \subseteq \mathcal{L}[\tau]$ we call T an *$\mathcal{L}$-theory.* A logic further has a satisfaction relation $\models_{\mathcal{L}}$, possibly holding between τ-structures $\mathcal{A}$ and some $\varphi \in \mathcal{L}[\tau]$. Importantly, $\models_{\mathcal{L}}$ is defined by some formula in the language of set theory, possibly with parameters. If $\mathcal{A} \models_{\mathcal{L}} \varphi$ for all $\varphi \in T \subseteq \mathcal{L}[\tau]$, then $\mathcal{L}$ is a *model* of the $\mathcal{L}$-theory T. We further assume that if $f : \tau \to \sigma$ is a renaming, then f induces a bijection, also called a *renaming*,

$f : \mathcal{L}[\tau] \to \mathcal{L}[\sigma]$ such that for any τ-structure $\mathcal{A}$: $\mathcal{A} \models_{\mathcal{L}} \varphi$ iff $f(\mathcal{A}) \models_{\mathcal{L}} f(\varphi)$. If $T \subseteq \mathcal{L}[\tau]$ is an $\mathcal{L}$-theory, we call $f``T$ a *copy* of T. We say that T is *$<\kappa$-satisfiable* for some cardinal κ, if every $T_0 \in \mathcal{P}_\kappa T$ has a model.

The concrete logics we will consider are second-order logic $\mathcal{L}^2$, as well as infinitary versions of second order logic $\mathcal{L}^2_{\kappa\lambda}$ for regular cardinals $\kappa \geq \lambda$, which allows for conjunctions and disjunctions over sets of formulas of size $< \kappa$ and quantification over strings of (first- or second-order) variables of length $< \lambda$. Recall that there is a sentence Φ of second-order logic, known as Magidor's Φ, such that $(M, E) \models \Phi$ iff $M \cong V_\alpha$ for some ordinal α (cf. [14]; the original construction requires that α is a limit ordinal; the general case is an easy adaptation). For later purposes, fix a large finite fragment ZFC* of ZFC, which ZFC proves to be satisfied in the V_α for α any limit ordinal and which is, in particular, large enough to prove that the universe is the union of the rank-initial segments V_α and that Φ is true in precisely those structures isomorphic to some V_α.

We fill further consider *sort logics*, an expansion of second-order logic introduced by Väänänen (cf. [18, 19] for details). The main feature of sort logics are *sort quantifiers* written as $\tilde{\exists}$ and $\tilde{\forall}$. A formula $\tilde{\exists} X \varphi(X)$ involving a sort quantifier over some relation variable X of arity n is true in a structure $\mathcal{A}$ iff $\mathcal{A}$ can be expanded by an additional domain B such that there is a subset $Y \subseteq B^n$ such that the expanded structure satisfies the formula $\varphi(B)$, i.e., the sort quantifiers search outside the structure itself, ranging over the whole universe V, for sets that satisfy some relation described by $\varphi(X)$. Because we would run into definability of truth issues otherwise, sort logics are graded into $\mathcal{L}^{s,n}$ by the natural numbers n. A sentence of $\mathcal{L}^{s,n}$ is only allowed to include n-alternations of sort quantifiers $\tilde{\exists}$ and $\tilde{\forall}$. We will in particular consider infinitary sort logics $\mathcal{L}^{s,n}_{\kappa\omega}$ which expand $\mathcal{L}^{s,n}$ by conjunctions and disjunctions of size $< \kappa$. We require that the syntax of vocabularies, $\mathcal{L}^2_{\kappa\lambda}$ and, $\mathcal{L}^{s,n}_{\kappa\omega}$ is coded in some reasonable way. More precisely, if $j : V \to M$ is some elementary embedding with $\mathrm{crit}(j) \geq \kappa$, then we require that for any vocabuly τ, j restricts to a renaming $j : \tau \to j``\tau$, and if $T \subseteq \mathcal{L}^2_{\kappa\lambda}$, then $j``\mathrm{T} \subseteq \mathcal{L}^2_{\kappa\lambda}$ is a copy of T, and analogously for $\mathcal{L}^{s,n}_{\kappa\omega}$.

2.3 Weak and strong Henkin models

Recall the Henkin semantics for second-order logic $\mathcal{L}^2$. If $\varphi \in \mathcal{L}^2[\tau]$ is a second-order sentence, some pair $(\mathcal{A}, P)$ consisting of a τ-structure $\mathcal{A}$ and $P \subseteq \mathcal{P}(A)$ is a (classical) *Henkin model* of φ, if $\mathcal{A}$ is seen to satisfy φ if we let the second-order quantifiers appearing in φ run over P (as opposed to the full power set of A). Notice that if M is some transitive set such that $\mathcal{A}$, $\varphi \in M$, and further $M \models$ "$\mathcal{A} \models_{\mathcal{L}^2} \varphi$", then $(\mathcal{A}, \mathcal{P}^M(A))$ is a Henkin model of φ, i.e., being a Henkin model is similar to evaluating the truth of "$\mathcal{A} \models_{\mathcal{L}^2} \varphi$" in some model of set theory that does not have A's full power set.

This served as motivation in [9] to generalise the notion of Henkin model to general logics in the following way. We present a simplified version of the notion considered there.

Definition 2. Let $\mathcal{L}$ be a logic, τ a vocabulary, $T \subseteq \mathcal{L}[\tau]$, M a transitive set and $\mathcal{A} \in M$. Then the pair $(M, \mathcal{A})$ is a called a *weak $\mathcal{L}$-Henkin model of* T iff there is a copy T^* of T such that for any $\varphi \in T^*$, we have $(M, \in) \models$ "$\mathcal{A} \models_{\mathcal{L}} \varphi$".

The main difference to the definition given in [9] is, that there the authors demand that the transitive set appearing in their corresponding definition of a Henkin model satisfies (some fragment of) ZFC. In practice, we also want weak Henkin models to satisfy some amount of ZFC. However, it depends on the context which fragment of ZFC is the appropriate one. We therefore decided to outsource fixing the "right" fragment of ZFC to the statement of our theorems. Further (minor) differences are that we explicitly work with models of copies of theories, and that we demand the existence of a single structure $\mathcal{A} \in M$ such that all $\varphi \in T^*$ are satisfied by $\mathcal{A}$ when computing satisfaction in M, while they work with a coherent system of structures $\mathcal{A}_\varphi$, all defined on the same set A, with $M \models$ "$\mathcal{A}_\varphi \models_\mathcal{L} \varphi$". Note however that we do *not* demand that $\mathcal{A}$ is a τ^*-structure for τ^* a renamed version of τ. Instead, $\mathcal{A}$ may also be a σ-structure for some $\sigma \supseteq \tau^*$.

We give the notion the qualification "weak" Henkin model to distinguish it from the stronger notion we will introduce below. In [9, Theorem 3.6], weak Henkin models were used to give a compactness characterisation of Woodin cardinals. Boney in [8, Theorem 4.7] used classical Henkin models to give a characterisation of strong cardinals. Referring to weak Henkin models, his result can be stated as follows:

Theorem 3 (Boney; [8, Theorem 4.7])**.** The following are equivalent for a cardinal κ:

(1) κ is strong.
(2) For any $\lambda > \kappa$ and any theory $T \subseteq \mathcal{L}^2_{\kappa\omega}$ that can be written as an increasing union $T = \bigcup_{\alpha\in\kappa} T_\alpha$ of theories T_α which each have a model of size $\geq \kappa$, there is a weak Henkin model $(M, \mathcal{A})$ of T such that $V_\lambda \subseteq M$, $M \models \mathrm{ZFC}^*$ and $|A| \geq \lambda$.

The notion of weak Henkin model $(M, \mathcal{A})$ of a theory T has some unexpected features. The set M does not need to contain T, but it is from the outside that we see that $M \models$ "$\mathcal{A} \models \varphi$" for every φ in (a copy of) T. If we require that M contains T, we get a stronger notion.

Definition 4. Let $\mathcal{L}$ be a logic, τ a vocabulary, $T \subseteq \mathcal{L}[\tau]$, M a transitive set such that $T \in M$, and $\mathcal{A} \in M$. Then the pair $(M, \mathcal{A})$ is called a *strong $\mathcal{L}$-Henkin model of* T iff $(M, \in) \models$"$\mathcal{A} \models_\mathcal{L} T$".[1]

This notion will be used in §§ 3 and 4 to characterise supercompact and $\mathrm{C}^{(n)}$-extendible cardinals, respectively.

3 Supercompact cardinals

For our characterisation, recall that $\mathcal{L}_{\kappa\omega}$ can define all ordinals $< \kappa$.

Theorem 5. The following are equivalent for a cardinal κ:

(1) κ is supercompact.
(2) For every λ, if $T \subseteq \mathcal{L}^2 \cup \mathcal{L}_{\kappa\omega}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^2 \cup \mathcal{L}_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*$ and $V_\lambda \subseteq M$.
(3) For every λ, if $T \subseteq \mathcal{L}^2_{\kappa\omega}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^2_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*$ and $V_\lambda \subseteq M$.
(4) For every λ, if $T \subseteq \mathcal{L}^2_{\kappa\kappa}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^2_{\kappa\kappa}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*$ and $V_\lambda \subseteq M$ and $M^\lambda \subseteq M$.

Proof. Clearly (4) implies (3), and (3) implies (2). We first show that (1) implies (4). So let T be a $<\kappa$-satisfiable $\mathcal{L}^2_{\kappa\kappa}$-theory. Take a $\beth$-fixed point λ of cofinality at least κ large enough such that $T \in V_\lambda$ and V_λ has a model for every $< \kappa$-sized subset of T. By supercompactness, let $j : V \to M$ be elementary with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$. Note that the restriction $i = j \restriction V_\lambda : V_\lambda \to V^M_{j(\lambda)}$ is an elementary embedding, and this implies $V^M_{j(\lambda)} \models \mathrm{ZFC}^*$. Because V_λ believes that T is $<\kappa$-satisfiable, by elementarity $V^M_{j(\lambda)} \models$"$i(T)$ is $< i(\kappa)$ satisfiable". By closure of M, we get that $i``T \in M$ and thus that $i``T \in V^M_{j(\lambda)}$. Further $|i``T|^{V^M_{j(\lambda)}} = |i``T|^M < \lambda < i(\kappa)$. Thus, $V^M_{j(\lambda)}$ believes that there is a model $\mathcal{B} \models i``T$. Because $\mathrm{crit}(i) = \kappa$, with the renaming $i : \tau \to i``\tau$ we have that $i``T$ is a copy of $T$. Notice that by closure under $\lambda$-sequences, $M$ and hence $V^M_{j(\lambda)}$ knows about the renamings $i : \tau \to i``\tau$ and $i : T \to i``T$. We can therefore also rename $\mathcal{B}$ in $V^M_{j(\lambda)}$ to a τ-structure $\mathcal{A}$, which

[1] We would like to thank the anonymous referee for their questions about an earlier version of this article, which lead us to formulate this definition in its current form.

$V^M_{j(\lambda)}$ believes to satisfy T. Notice that $V_\lambda \subseteq V^M_{j(\lambda)}$. Further, $\mathrm{cof}(j(\lambda))^M \geq j(\kappa) > \lambda > \kappa$. By closure of M, this implies that $V^M_{j(\lambda)}$ is λ-closed. Summarising, $(V^M_{j(\lambda)}, \mathcal{A})$ is a strong Henkin model as desired.

And now assume (2) and let us show (1). Take a cardinal $\lambda > \kappa$ of $\mathrm{cof}(\lambda) \geq \kappa$. Consider the theory

$$T = \mathrm{ElDiag}_{\mathcal{L}^2 \cup \mathcal{L}_{\kappa\omega}}(V_{\lambda+1}, \in) \cup \{c_i \in d \wedge |d| < c_\kappa \colon i < \lambda\},$$

where d is a new constant and the c_i are the constants used in the elementary diagram. If $T_0 \subseteq T$ is of size $< \kappa$, there is $X \subseteq \lambda$ such that $|X| < \kappa$ and the sentence "$c_i \in d \wedge |d| < c_\kappa$" is contained in T_0 iff $i \in X$. Then letting d be interpreted by X, we get that $(V_{\lambda+1}, \in, d)$ witnesses that T_0 is satisfiable. So by (2), we get a transitive model M of ZFC* such that $V_\alpha \subseteq M$ for some large $\alpha > \lambda$ and $\mathcal{A} \in M$ such that $M \models$"$\mathcal{A} \models T$". We may take α large enough such that $T \in V_\alpha$. Notice that T is a theory in a language $\tau \in \{\in, c_x, d \colon x \in V_{\lambda+1}\}$. Because with T, also $\tau \in V_\alpha$, and thus also the structure $N = (V_{\lambda+1}, \in, c^N_x)_{x \in V_{\lambda+1}}$ in which every c_x is interpreted by x itself, and which witnesses that $(V_{\lambda+1}, \in)$ satisfies its own elementary diagram, is in V_α and hence in M. Because first-order satisfaction is absolute between M and V, M understands that T contains the elementary diagram of $(V_{\lambda+1}, \in)$ and therefore believes that there is an elementary embedding $j : V_{\lambda+1} \to \mathcal{A}$. Again, by absoluteness of first-order satisfaction, this is really an elementary embedding. Because $M \models$ ZFC* and T contains Magidor's Φ, M believes $\mathcal{A}$ to be some rank-initial segment and so we have to have $A = V^M_{\beta+1}$ for some β. Because $c^{\mathcal{A}}_i \in d^{\mathcal{A}}$ for every $i < \lambda$, we get that $j(\kappa) > |d|^{\mathcal{A}} \geq \lambda$. In particular, $\mathrm{crit}(j) \leq \kappa$. Because also $\mathcal{L}_{\kappa\omega}$-satisfaction is absolute for transitive models and $\mathcal{L}_{\kappa\omega}$ can define all ordinals $< \kappa$, those have to be fixed by j. Thus $\mathrm{crit}(j) = \kappa$. Notice that $j``\lambda$ is definable from $j$ and $\lambda$ and so $j``\lambda \in M$ and therefore in $V^M_{\beta+1}$. Summarising, we have an elementary embedding $j : V_{\lambda+1} \to V^M_{\beta+1}$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j``\lambda \in V^M_{\beta+1}$. We can therefore let, for $X \subseteq \mathcal{P}_\kappa\lambda$:

$$X \in U \text{ iff } j``\lambda \in j(X).$$

It is standard to check that this defines a fine, normal and κ-complete ultrafilter U over $\mathcal{P}_\kappa\lambda$. To check normality, for example, if f is a regressive function on $\mathcal{P}_\kappa\lambda$ and so $\{s \in \mathcal{P}_\kappa\lambda \colon f(s) \in s\} \in U$. Then $j``\lambda \in \{s \in \mathcal{P}_{j(\kappa)}j(\lambda) \colon j(f)(s) \in s\}$ and hence $j(f)(j``\lambda) = j(\gamma)$ for some $\gamma < \lambda$. Therefore $\{s \in \mathcal{P}_\kappa\lambda \colon f(s) = \gamma\} \in U$. Hence κ is λ-supercompact for arbitrarily large λ. □

Note that in (3), M is closed under λ-sequences. In this context, the relevant closure is (the implied) closure under κ-sequences, as this makes M correct about $\mathcal{L}_{\kappa\kappa}$-satisfaction.

We would like to make some remarks about related results. An argument by Dimopoulos shows that if κ is strong and strongly compact then κ is also *jointly strong and strongly compact*, i.e., for every λ there is an embedding simultaneously witnessing κ being λ-strong and λ-strongly compact (cf. [10, Proposition 2.3]). And Apter and Hamkins show that it is consistent to have a cardinal which is both strong and strongly compact, but not supercompact (cf. [2, Theorem 1.2]). Boney points out, that in his framework considering classical Henkin models for second-order logic, a compactness principle using full compactness instead of the chain compactness property from Theorem 3 characterises cardinals which are jointly strong and strongly compact (cf. [8, p. 159]), which by Dimopoulos' result comes down to κ being strong and strongly compact. This translates to the terminology of weak Henkin models, i.e., considering weak Henkin models instead of strong ones in Theorem 5 characterises that κ is strong and strongly compact. By the result of Apter and Hamkins the latter is not equivalent to supercompactness of κ. In particular, this shows that usage of strong Henkin models is necessary for the result of Theorem 5 in the following way.

Theorem 6. It is consistent that κ is not supercompact, but that for every λ, if $T \subseteq \mathcal{L}^2_{\kappa\omega}$ is a $<\kappa$-satisfiable theory, then there is a weak $\mathcal{L}^2_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models$ ZFC* and $V_\lambda \subseteq M$.

4 $\mathrm{C}^{(n)}$-extendible cardinals and Vopěnka's Principle

To provide some background, we mention that model theory of extensions of first-order logic has close connections to VP: a cardinal κ is called the *compactness number* of a logic $\mathcal{L}$, if it is the smallest cardinal such that any theory $T \subseteq \mathcal{L}$ is satisfiable, provided all $T_0 \in \mathcal{P}_\kappa T$ are satisfiable.

Theorem 7 (Makowsky; [16, Theorem 2])**.** VP holds if and only if every logic has a compactness number.

Our result will be the analogue for strong Henkin compactness of all logics, making use of Bagaria's stratification of VP in terms of $\mathrm{C}^{(n)}$-extendible cardinals (Theorem 1). In our proof, we will use a number of results from the literature.

Theorem 8 (Bagaria & Goldberg; [6, Theorem 2.6])**.** The following are equivalent for every $n \geq 1$ and every cardinal κ:

(1) κ is $\mathrm{C}^{(n)}$-extendible.
(2) For every $\lambda > \kappa$, $\lambda \in \mathrm{C}^{(n+1)}$, there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $M \models$"$\lambda \in \mathrm{C}^{(n+1)}$".
(3) For every $\lambda > \kappa$, $\lambda \in \mathrm{C}^{(n+1)}$, there is a fine, normal and κ-complete ultrafilter U on $\mathcal{P}_\kappa\lambda$ such that $\{s \in \mathcal{P}_\kappa\lambda \colon \mathrm{ot}(s) \in \mathrm{C}^{(n+1)}\} \in U$.

We remark that this theorem of Bagaria & Goldberg shows that extendibles and $\mathrm{C}^{(n)}$-extendibles are direct strengthenings of supercompact cardinals.

Proposition 9 (Folklore; cf., e.g., [9, Proposition 2.2])**.** For every n, $\mathcal{L}^{s,n}$ has a sentence $\Phi^{(n)}$ such that $(M, E) \models \Phi^{(n)}$ iff $(M, E) \cong (V_\alpha, \in)$ for some $\alpha \in \mathrm{C}^{(n)}$.

For any natural number n, let us fix a finite fragment ZFC^*_n of ZFC, expanding ZFC^* and sufficiently large to prove that $\Phi^{(n)}$ axiomatizes the class of models isomorphic to some $(V_\alpha, \in)$ such that $\alpha \in \mathrm{C}^{(n)}$.

Theorem 10. The following are equivalent for every $n \geq 1$ and every cardinal κ:

(1) κ is $\mathrm{C}^{(n)}$-extendible.
(2) For every $\lambda \in \mathrm{C}^{(n+1)}$, if $T \subseteq \mathcal{L}^{s,n+1} \cup \mathcal{L}_{\kappa\omega}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^{s,n+1} \cup \mathcal{L}_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*_{n+1}$ and $V_\lambda \prec_{\Sigma_{n+1}} M$.
(3) For every $\lambda \in \mathrm{C}^{(n+1)}$, if $T \subseteq \mathcal{L}^{s,n+1}_{\kappa\omega}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^{s,n+1}_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*_{n+1}$ and $V_\lambda \prec_{\Sigma_{n+1}} M$.

Proof. The proof proceeds similar to the supercompactness case. Clearly (3) implies (2). Assume (1) and let us show (3). Let T be $<\kappa$-satisfiable over the vocabulary τ. By the reflection theorem, take $\lambda = \beth_\lambda \in \mathrm{C}^{(n+1)}$ such that V_λ satisfies ZFC^*_{n+1} and large enough such that V_λ verifies that T is $<\kappa$-satisfiable. Take $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $M^\lambda \subseteq M$ and $M \models$ "$\lambda \in \mathrm{C}^{(n+1)}$". Again $i = j \restriction V_\lambda : V_\lambda \to V^M_{j(\lambda)}$ is elementary. In particular, $V^M_{j(\lambda)} \models \mathrm{ZFC}^*_{n+1}$; further $V^M_{j(\lambda)} \models$"$i(T)$ is $< i(\kappa)$ satisfiable" and so $V^M_{j(\lambda)}$ has a model $\mathcal{B}$ for the copy i"T. As earlier, by closure of M, this can be renamed to a τ-structure $\mathcal{A} \in V^M_{j(\lambda)}$ which $V^M_{j(\lambda)}$ believes to satisfy T. Again, $V_\lambda \subseteq V^M_{j(\lambda)}$. Finally, because $\lambda \in \mathrm{C}^{(n+1)}$, by elementarity of j, we have $M \models j(\lambda) \in \mathrm{C}^{(n+1)}$. Thus $V^M_{j(\lambda)} \prec_{\Sigma_{n+1}} M$. Also by assumption $M \models$ "$\lambda \in \mathrm{C}^{(n+1)}$" and so $V_\lambda \prec_{\Sigma_{n+1}} M$. Because $V_\lambda \subseteq V^M_{j(\lambda)}$, this together implies $V_\lambda \prec_{\Sigma_{n+1}} V^M_{j(\lambda)}$. Summarising, $(V^M_{j(\lambda)}, \mathcal{A})$ is a strong Henkin model as desired.

Now assume (2) and let us show (1). Let $\lambda > \kappa$ be in $\mathrm{C}^{(n+1)}$ and of cofinality $\mathrm{cof}(\lambda) \geq \kappa$. Consider

$$T = \mathrm{ElDiag}_{\mathcal{L}^{s,n+1} \cup \mathcal{L}_{\kappa\omega}}(V_{\lambda+1}, \in) \cup \{c_i \in d \wedge |d| < c_\kappa \colon i < \lambda\}.$$

Again, for $<\kappa$-sized subsets of T, we can get a model by considering $V_{\lambda+1}$ itself. So for some $\alpha \in C^{(n+1)}$ much greater than λ and such that $T \in V_\alpha$, by assumption we get an $M \models \mathrm{ZFC}^*_{n+1}$ such that $V_\alpha \prec_{\Sigma_{n+1}} M$ and there is $\mathcal{A} \in M$ which M believes to be a model of T. As before, M has a first-order elementary embedding $j : V_{\lambda+1} \to \mathcal{A}$. By Magidor's Φ, we have $A = V^M_{\beta+1}$ for some β. Further, because $\lambda \in \mathrm{C}^{(n+1)}$, T contains a sentence coding that $\Phi^{(n+1)}$ (cf. Proposition 9) holds in V_λ, i.e., in the rank initial segment cut off at the largest ordinal λ of $V_{\lambda+1}$. Then M believes that this sentence holds in $V^M_{\beta+1}$ and so that V^M_β satisfies $\Phi^{(n+1)}$. Since $M \models \mathrm{ZFC}^*_{n+1}$, thus $M \models$ "$\beta \in \mathrm{C}^{(n+1)}$". Again, our theory implies that $j(\kappa) > \lambda$ and because $j``\lambda$ is definable in $M$, we have $j``\lambda \in V^M_{\beta+1}$. Summarising, we have an elementary embedding $j : V_{\lambda+1} \to V^M_{\beta+1}$ with $j(\kappa) > \lambda$ and $j``\lambda \in V^M_{\beta+1}$. Define a fine, normal and $\kappa$-complete ultrafilter on $\mathcal{P}_\kappa\lambda$ as usual, by letting $X \in U$ iff $j``\lambda \in j(X)$. By Bagaria's and Goldberg's Theorem 8, it suffices to verify that $X = \{s \in \mathcal{P}_\kappa\lambda \colon \mathrm{ot}(s) \in \mathrm{C}^{(n+1)}\} \in U$. Notice that because $\lambda \in \mathrm{C}^{(n+1)}$ and $\mathrm{cof}(\lambda) \geq \kappa$, for $s \in \mathcal{P}_\kappa\lambda$ we have

$$V_{\lambda+1} \models \forall s \in \mathcal{P}_\kappa\lambda(s \in X \leftrightarrow V_\lambda \models \text{“}\mathrm{ot}(s) \in \mathrm{C}^{(n+1)}\text{”}).$$

By elementarity,

$$V^M_{\beta+1} \models \forall s \in \mathcal{P}_{j(\kappa)}j(\lambda)(s \in j(X) \leftrightarrow V^M_\beta \models \text{“}\mathrm{ot}(s) \in \mathrm{C}^{(n+1)})\text{”}.$$

So we have to show that $V^M_\beta \models$ "$\lambda = \mathrm{ot}(j``\lambda) \in \mathrm{C}^{(n+1)}$". Because $M \models$"$\beta \in \mathrm{C}^{(n+1)}$", this is equivalent to $M \models$"$\lambda \in \mathrm{C}^{(n+1)}$". As really $\alpha \in \mathrm{C}^{(n+1)}$, and $\alpha > \lambda \in \mathrm{C}^{(n+1)}$, we have $V_\alpha \models$ "$\lambda \in \mathrm{C}^{(n+1)}$". Because $V_\alpha \prec_{\Sigma_{n+1}} M$ by assumption, this implies $M \models$ "$\lambda \in \mathrm{C}^{(n+1)}$", verifying $X \in U$. □

An easy adaptation of the above proof further gives:

Theorem 11. The following are equivalent for every $n \geq 1$ and every cardinal κ:

(1) κ is the smallest $\mathrm{C}^{(n)}$-extendible cardinal.
(2) κ is the smallest cardinal such that for every $\lambda \in \mathrm{C}^{(n+1)}$, if $T \subseteq \mathcal{L}^{s,n+1}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^{s,n+1}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*_{n+1}$ and $V_\lambda \prec_{\Sigma_{n+1}} M$.

Because the strength of any logic is bounded by $\mathcal{L}^{s,n}_{\kappa\omega}$ for some n and κ, and because VP is equivalent to the existence of $\mathrm{C}^{(n)}$-extendible cardinals for any n, our results imply the following Makoswky-like characterisation of VP:

Corollary 12. The following are equivalent:

(1) VP
(2) For any logic $\mathcal{L}$ and any natural number n, there is a cardinal κ such that if $\lambda \in \mathrm{C}^{(n)}$ and $T \subseteq \mathcal{L}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}$-Henkin model $(M, \mathcal{A})$ of T such that $V_\lambda \prec_{\Sigma_n} M$.

We would like to state some remarks about closely related results.

Firstly, similar characterisations of VP have been obtained with other model-theoretic properties. E.g., Boney showed that the existence of a compactness number of $\mathcal{L}^{s,n}$ is equivalent to the existence of a $\mathrm{C}^{(n)}$-extendible cardinal (cf. [8, §4.1]). Stavi showed that VP is equivalent to the existence of *Löwenheim-Skolem-Tarski* numbers for every logic (cf. [15]). Gitman and the first author showed that the existence of an *upward Löwenheim-Skolem-Tarski number* of $\mathcal{L}^{s,n}$ is equivalent to the existence of a $\mathrm{C}^{(n)}$-extendible cardinal and that VP is equivalent to the existence of upward Löwenheim-Skolem-Tarski numbers for every logic (cf. [11]).

Also, there is a weakening of Vopěnka's Principle with a category-theoretic motivation due to Adámek, Rosický, and Trnková called *Weak Vopěnka's Principle* WVP (cf. [1]) that has been stratified in a similar way by the notions of Π_n-strong cardinals by Bagaria and Wilson.

Theorem 13 (Bagaria & Wilson; [5, §5]). WVP holds if and only if for every n, there is a Π_n-strong cardinal.

Boney and the first author provided the following characterisation of Π_n-strong cardinals in terms of *weak* Henkin models of $\mathcal{L}^{s,n}$, published in the first author's Ph.D. thesis.

Theorem 14 (Boney & O.; [17, Theorem 2.3.6]). The following are equivalent for every $n \geq 2$ and every cardinal κ:

(1) κ is Π_n-strong
(2) For every λ which is a limit of $\mathrm{C}^{(n)}$ and every theory $T \subseteq \mathcal{L}^{s,n}_{\kappa\omega}$ that can be written as an increasing union $T = \bigcup_{\alpha<\kappa} T_\alpha$ of theories T_α that each have models of size $\geq \kappa$, there is a weak $\mathcal{L}^{s,n}_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*_n$, $|A| \geq \lambda$ and $V_\lambda \prec_{\Sigma_n} M$.

Theorems 13 & 14 together yield a characterisation of WVP in terms of weak Henkin models. One can therefore jump between the stratifications of VP by $\mathrm{C}^{(n)}$-extendible cardinals, and of its weakening WVP by Π_n-strong cardinals, by switching between assuming the compactness principle about *strong* Henkin models from Theorem 10, and the compactness principle about *weak* Henkin models from Theorem 14.

5 Superstrong Cardinals

We would like to close by showing how superstrong cardinals can be characterised via compactness properties for weak Henkin models. To our best knowledge, this is the first known model-theoretic characterisation of superstrong cardinals.

Theorem 15. The following are equivalent:

(1) κ is superstrong with target λ.
(2) For any theory $T \subseteq \mathcal{L}^2_{\kappa\omega}$ such that $\mathrm{rk}(T) < \kappa + \omega$ and that can be written as an increasing union $T = \bigcup_{\alpha\in\kappa} T_\alpha$ of theories T_α which each have a model of rank $< \kappa + \omega$ and of size $\geq \kappa$, there is a weak $\mathcal{L}^2_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $V_\lambda \subseteq M \subseteq V_{\lambda+\omega}$, $M \models \mathrm{ZFC}^*$, $|A| \geq \lambda$ and $M \models \lambda = \beth^M_\lambda$.

Proof. First assume (1) and suppose we have a setup as in (2). Then there is a function f with domain κ such that $f(\alpha) \models T_\alpha$, $\mathrm{rk}(f(\alpha)) < \kappa + \omega$ and $|f(\alpha)| \geq \kappa$. Take an elementary embedding $j : V \to N$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $V_{j(\kappa)} \subseteq N$. Consider the sequence $(T_\alpha \colon \alpha < \kappa)$. Evaluating it via j leads to a sequence $j((T_\alpha \colon \alpha < \kappa)) = (T^*_\alpha \colon \alpha < j(\kappa))$ such that $j``T \subseteq T^*_\kappa$. Notice that $j``T$ is a copy of T. By elementarity, in N, we have that $j(f)(\kappa) \models T^*_\kappa$ and so in particular, for every $\varphi \in j``T$, $N \models$"$j(f)(\kappa) \models \varphi$". Further, $\mathrm{rk}(j(f)(\kappa)) < j(\kappa) + \omega = \lambda + \omega$ and thus $j(f)(\kappa) \in V^N_{\lambda+\omega}$. Then $(M, \mathcal{A}) = (V^N_{\lambda+\omega}, j(f)(\kappa))$ gives our desired Henkin model: Because $V_\lambda \subseteq N$, we have $V_\lambda \subseteq V^N_{\lambda+\omega} \subseteq V_{\lambda+\omega}$. As ZFC proves that ZFC^* holds in the limit stages of the cumulative hierarchy, $V^N_{\lambda+\omega} \models \mathrm{ZFC}^*$. Because $V^N_{\lambda+\omega}$ and N agree on second-order satisfaction, we have $V^N_{\lambda+\omega} = M \models$"$\mathcal{A} \models \varphi$" for every $\varphi \in j``T$. By elementarity, N, and hence M believes that $j(\kappa) = \lambda$ is a $\beth$-fixed point. Finally, note that λ is actually a (strong limit) cardinal as the target of a superstrong embedding, and so because by elementarity $N \models |j(f)(\kappa)| \geq j(\kappa) = \lambda$, that $|j(f)(\kappa)| \geq \lambda$ really holds in V.

And now assume (2). We show that κ is superstrong with target λ. By standard results (cf., e.g., [13, §26]), if $j : V_{\kappa+1} \to N$ is an elementary embedding such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $V_{j(\kappa)} \subseteq N$, and we derive an extender by letting for $a \in [\lambda]^{<\omega}$ and $X \subseteq [\kappa]^{<\omega}$,

$X \in E_a$ iff $a \in j(X)$,

then the extender power of the universe witnesses that κ is superstrong with target λ. So it is sufficient to derive an embedding as above. For this, consider the following theory:

$$T = \mathrm{ElDiag}_{\mathcal{L}^2_{\kappa\omega}}(V_{\kappa+1}, \in) \cup \{c_i < c < c_\kappa : i < \kappa\},$$

where c is a new constant and the c_i are the constants from the elementary diagram. Clearly, T can be considered to have rank $< \kappa + \omega$ and can be written as an increasing union of length κ of theories T_α for $\alpha < \kappa$ by considering in T_α only those bits of the second part of T such that $i < \alpha$. Then $(V_{\kappa+1}, \in)$ gives a model of T_α of size $\geq \kappa$ and of rank $< \kappa + \omega$. By (2), we get a transitive set M and $\mathcal{A} \in M$ such that $M \models$"$\mathcal{A} \models \varphi$" for every φ from (a copy of) T and such that $V_\lambda \subseteq M \subseteq V_{\lambda+\omega}$, $M \models \mathrm{ZFC}^*$, $|A| \geq \lambda$ and $M \models \lambda = \beth^M_\lambda$. Because T contains Magidor's Φ, we have that $A = V^M_\beta$ for some β. By size of A and $\lambda = \beth^M_\lambda$, we get $\beta \geq \lambda$. Further, by absoluteness of $\mathcal{L}_{\kappa\omega}$-satisfaction, in V we see that $\mathcal{A} \models \mathrm{ElDiag}_{\mathcal{L}_{\kappa\omega}}(V_{\kappa+1}, \in)$ and thus there is an elementary embedding $j : V_{\kappa+1} \to A = V^M_\beta$ such that $\mathrm{crit}(j) = \kappa$. Because $A \in M \subseteq V_{\lambda+\omega}$, this implies $\beta = \lambda + 1$ and then clearly $j(\kappa) = \lambda$. Because $V_\lambda \subseteq M$, finally $V_{j(\kappa)} = V_\lambda \subseteq V^M_{\lambda+1} = A$. □

Acknowledgements. The first author would like to acknowledge the support of Harvard University's Center of Mathematical Sciences and Applications and thank them for their hospitality during a visit in 2024 during which the results of this article were partly obtained.

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